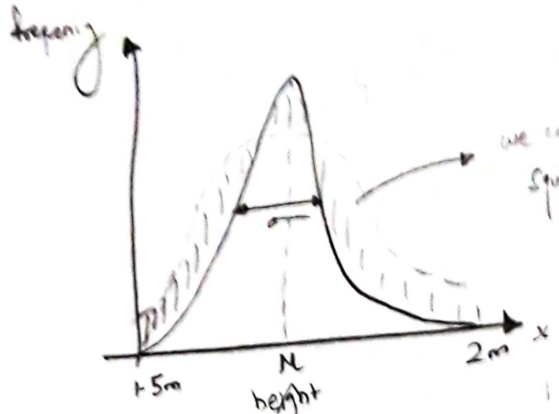
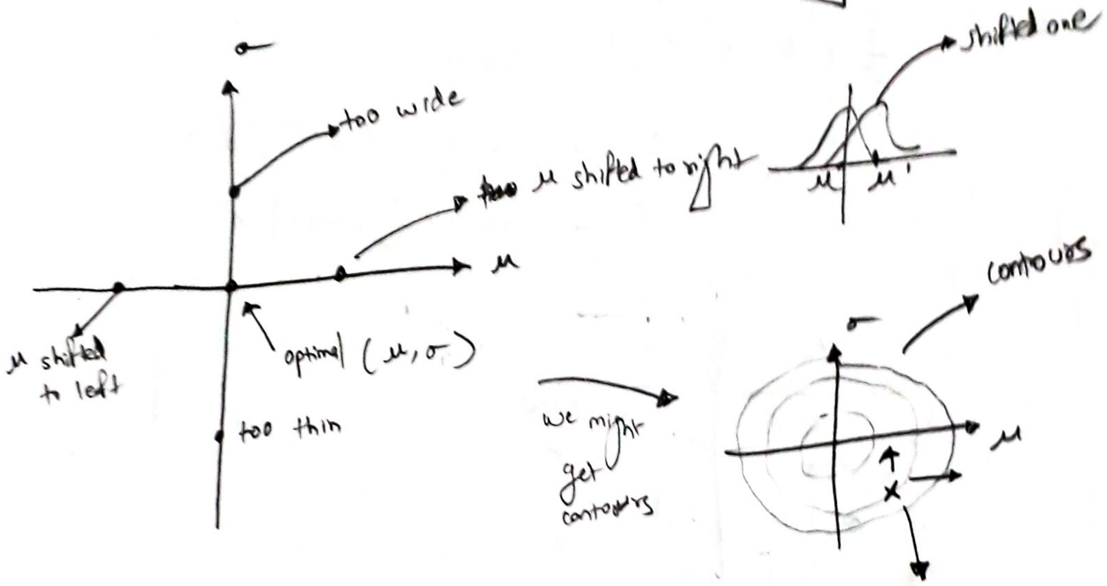
Linear algebra



we can add up or square the differences to get the goodness/badness of the fit

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right]$$



$$\begin{bmatrix} \mu \\ \sigma \end{bmatrix} \rightarrow \begin{bmatrix} \mu' \\ \sigma' \end{bmatrix}$$

(where can we go from x to get a better (μ, σ))

till optimal is found

"what we are trying to do is find a location in the space where the badness is minimized, the goodness is maximized and the function fits the data best."

interesting "The function we fit, has some parameters → (σ, μ) and we can plot the goodness of the fit (quality of the fit) varies as we vary those parameters

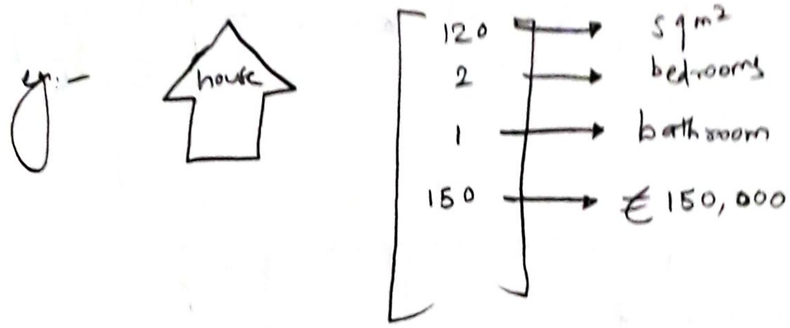
moves around these fitting parameter space are then just vectors"

Vector

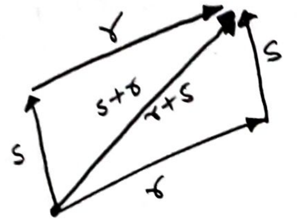
In physics → a vector is something that moves around space

In data science → a vector ~~contains~~ is something that describes an object.

eg:-

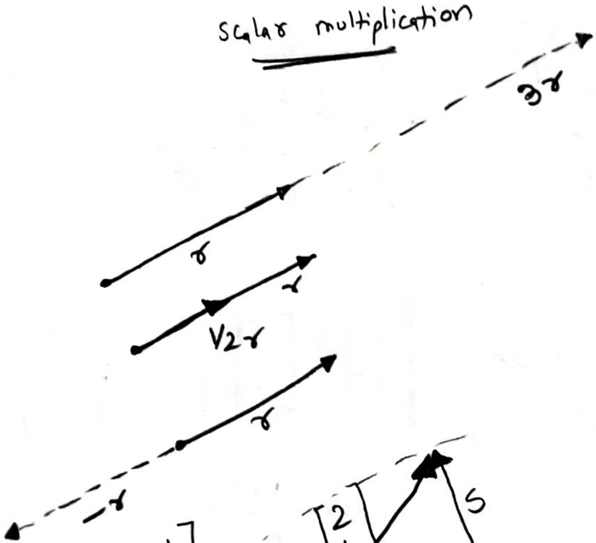


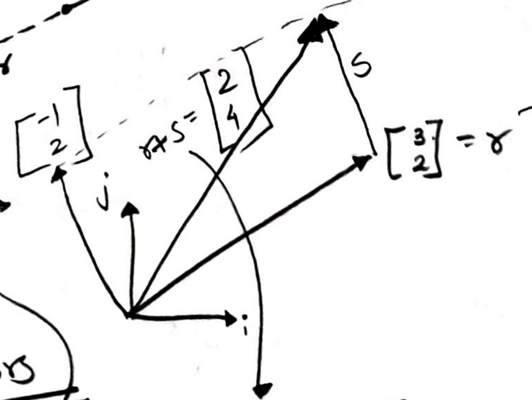
addition



$r+s = s+r$

scalar multiplication





coordinate system →

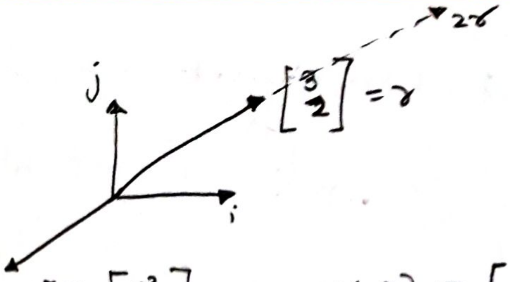
(i, j are **basis vectors**)

→ it means scalar multiplication of $3i$, $2j$ and adding them together. $(3i + 2j)$

to get $r+s$, we can just add up the components of 'r' and 's'

"Vector addition is **associative**"

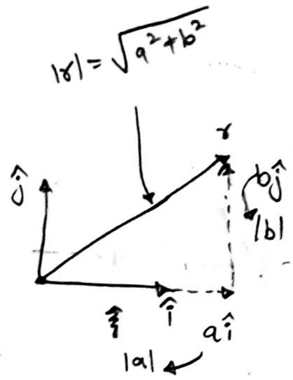
→ $(r+s)+t = r+(s+t)$



$-r = \begin{bmatrix}-3\\-2\end{bmatrix}$ $\quad r + (-r) = \begin{bmatrix}3 & -3\\ 2 & -2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

Vector (length and direction)

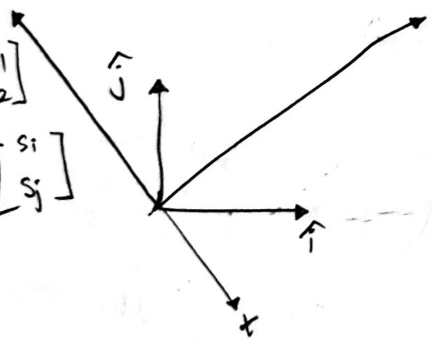
(notation :- $\left.\begin{matrix}\hat{i}\\ \hat{j}\end{matrix}\right\}$ → denotes they are of length 1)



$r = \begin{bmatrix}a\\b\end{bmatrix}$

$|r| = \sqrt{a^2+b^2}$

$s = \begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}s_i\\s_j\end{bmatrix}$



$r = \begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}r_i\\r_j\end{bmatrix}$

dot product

$$\begin{aligned} r \cdot s &= r_i s_i + r_j s_j \\ &= 3(-1) + 2(2) \\ &= 1 \\ &= s \cdot r \end{aligned}$$

This makes it commutative

It is also distributive over addition

$r \cdot (s+t) = r \cdot s + r \cdot t$

associative over scalar multiplication

$r \cdot (as) = a(r \cdot s)$

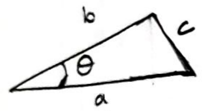
a constant

if we want to get the size of a vector :- (r)

$$r \cdot r = r_1 r_1 + r_2 r_2$$
$$= r_1^2 + r_2^2$$
$$= \left(\sqrt{r_1^2 + r_2^2}\right)^2$$
$$= |r|^2$$

$$|r| = \sqrt{r_1^2 + r_2^2} \qquad \text{if } r = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

Cosine rule



$c^2 = a^2 + b^2 - 2ab\cos\theta$

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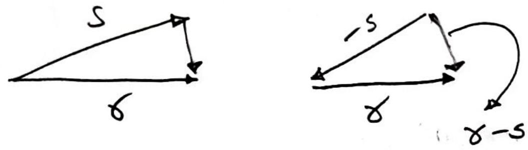


from cosine rule :-

$$|r-s|^2 = |r|^2 + |s|^2 - 2|r||s|\cos\theta \quad -1.$$

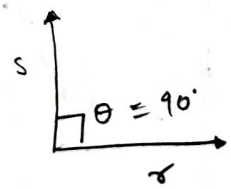
$$(r-s)\cdot(r-s) = r\cdot r - 2s\cdot r + s\cdot s \quad -2.$$
$$= |r|^2 - 2s\cdot r + |s|^2$$

subtracting (2) and (1)

we get :-

$$\underline{r\cdot s = |r||s|\cos\theta}$$

if :-

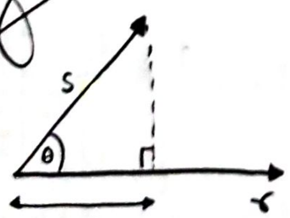


$\longrightarrow r\cdot s = |r||s| \times 0$

$r\cdot s = 0$

We say that both vectors are orthogonal to each other

~~inversely~~



$\cos\theta = \dfrac{adj}{hyp} = \dfrac{adj}{|s|}$

$$r\cdot s = |r| \underbrace{|s|\cos\theta}_{\text{adjacent}}$$

$|r| \cdot$ projection of s on r

$\therefore$ if s is at 90° angle from r, there would be no projection and $r\cdot s = 0$

$$\frac{r \cdot s}{|r|} = |s| \cdot \cos\theta \quad \text{(scalar projection)}$$

↓ a number as $r \cdot s$ and $|r|$ are no.s

$$\frac{r \cdot s}{|r| \cdot |r|} \cdot r$$

(vector projection)
↳ scalar projection also encoded with the direction of R

Problem:— $s = \begin{bmatrix} 10 \\ 5 \\ -6 \end{bmatrix}$ $r = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$

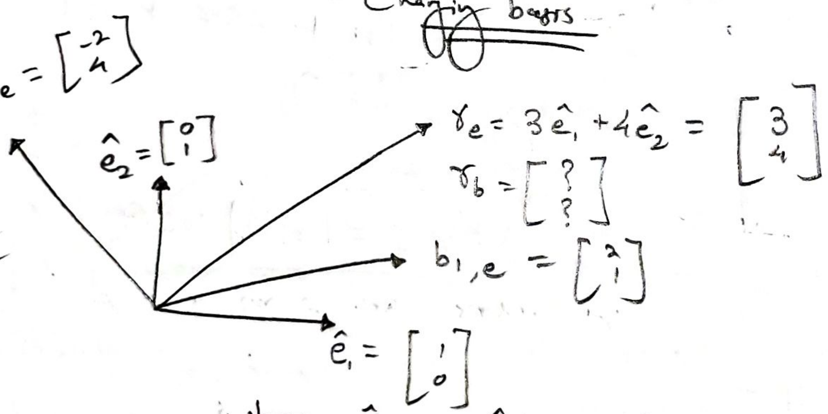
scalar projection of s onto r

$$\frac{30 - 20}{\sqrt{9+16}} = \frac{10}{\sqrt{25}} = \underline{\underline{2}}$$

Triangle inequality

$$|a+b| < |a| + |b|$$

Changing basis

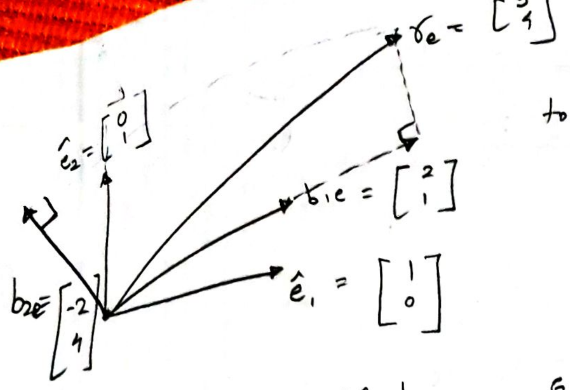


where $\hat{e}_1$ and $\hat{e}_2$ are basis vectors of unit length

interesting
If they are not, we need matrices to do transformation of axes from e to b

If b_1 and b_2 are at 90° from each other
↓
it is computationally faster and easier to find r_b

we will first check if $b_1 \cdot b_2$ is 0 (i.e. they are at 90° angle)
↳ $2 \times -2 + 4 \times 1 = \underline{0}$ (yes!)

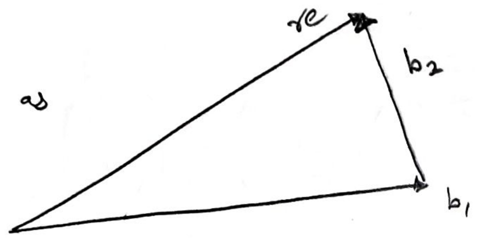


to find r_b

$$\frac{r_e \cdot b_1}{|b_1|^2} = \frac{6+4}{5} = 2\,b_1 = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\frac{r_e\, b_2}{|b_2|^2} = \frac{-6+16}{16+4} = \frac{1}{2}\,b_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

now, as



$r_e = b_1 + b_2$

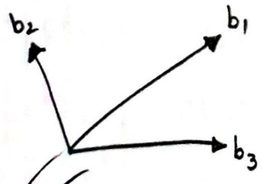
$\therefore\ r_e = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

which it is! in basis $\hat{e}$

$\therefore$ in basis b :- $r_b = \begin{bmatrix} 2 \\ 1/2 \end{bmatrix}$

" We can redescribe a vector using some other axis, some other basis vectors in the case, the new basis vectors are orthogonal to each other. "

<u>linear dependance/ linear independance</u>



$b_3 \neq a_1 b_1 + a_2 b_2$

linearly independent

~~linearly~~ linear independent is when b_3 cannot be represented using b_1 and b_2.

<u>If it</u> can, b_3 will be linearly dependant on b_1 and b_2

(i.e. b_3 does not lie in the plane spanned by b_1 and b_2)

{ Orthonormal vectors → 90° unit length
Orthogonal vectors → 90° to each other

Basis is a set of n vectors that:

(i.) are not linear combinations of each other
(are linearly independant)

(ii.) span the space
(the space is then n-dimentional)

Module 3

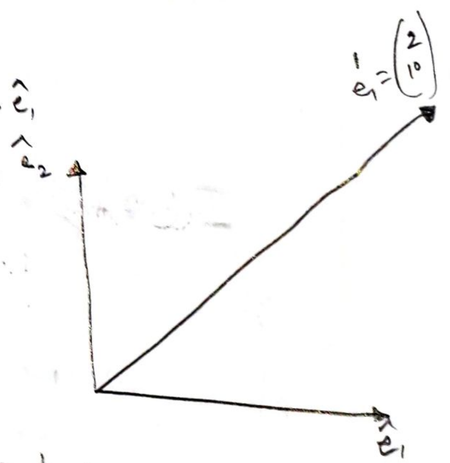
matrices and vectors

$2a + 3b = 8$
$10a + 1b = 13$

$$\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 8 \\ 13 \end{pmatrix}$$

eg:- if we take $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \hat{e}_1$

$$\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \end{pmatrix}$$



" So now we can see what we mean by linear algebra. Linear algebra is linear, because it takes input values, our a and b, and multiplies them by constants. So everything is linear.

And it's algebra, that it is a notation of mathematical objects and a system to manipulate those notations.

Linear algebra is a system of manipulating vectors in the space described by vectors. "

$$\begin{bmatrix} 7 & -6 \\ 12 & 8 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 7 & -6 \\ 12 & 8 \end{bmatrix}\left(5\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 5\left(\begin{bmatrix} 7 & -6 \\ 12 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + 6\left(\begin{bmatrix} 7 & -6 \\ 12 & 8 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= 5\begin{pmatrix} 7 \\ 12 \end{pmatrix} + 6\begin{pmatrix} -6 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} 35 - 36 \\ 60 + 48 \end{pmatrix} = \begin{pmatrix} -1 \\ 108 \end{pmatrix}$$

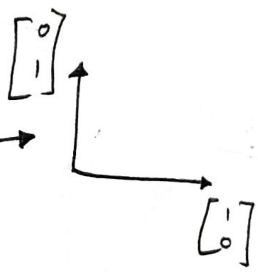
$$A r = r'$$

$$A(nr) = nr'$$

$$A(r+s) = Ar + As$$

Types of matrices

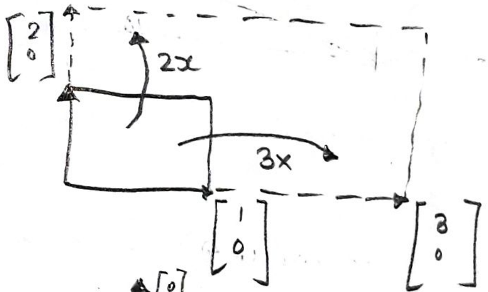
- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow$



I (identity matrix)

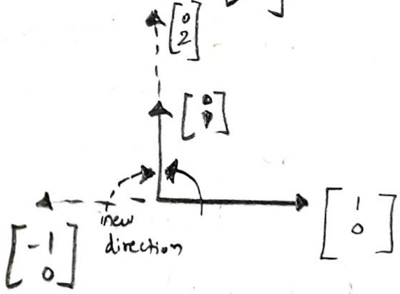
Stretches

- $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow$



Inversion

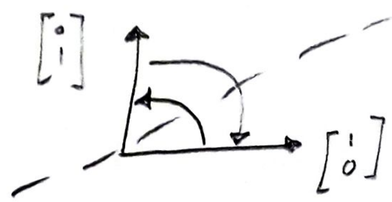
- $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow$



- $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (flips in both coordinates (inversion))

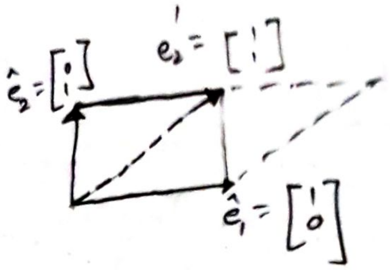
Mirrors

- $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} \rightarrow$ like a mirror

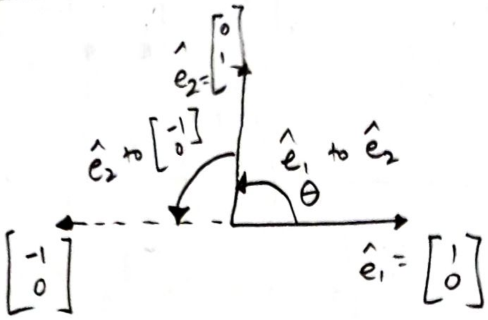


shears

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$



rotation

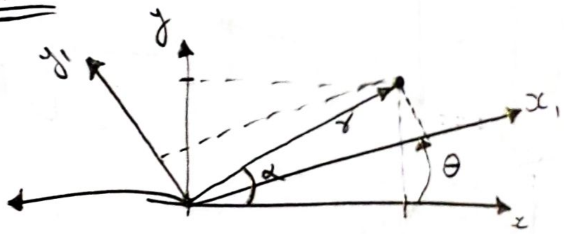


in this case done using ↺ $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

general rotation in 2D (↻ +ve) = $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

derivation

ORIGIN IS FIXED DURING ROTATION



$x = r\cos\alpha$

$y = r\sin\alpha$

$x_1 = r\cos(\alpha - \theta) = r\cos\alpha\cos\theta + r\sin\alpha\sin\theta$

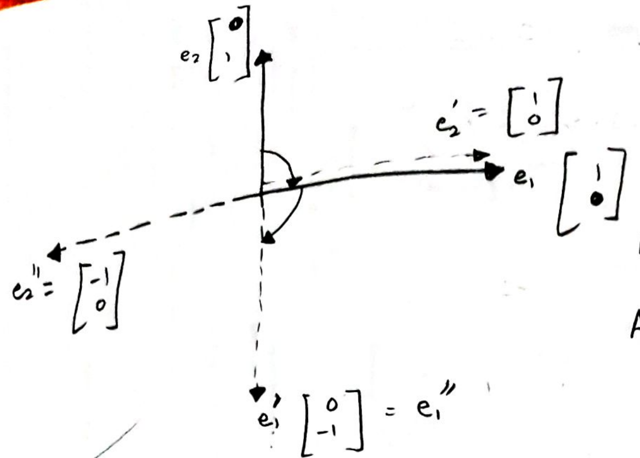
$y_1 = r\sin(\alpha - \theta) = r\sin\alpha\cos\theta - r\cos\alpha\sin\theta$

$x_1 = x\cos\theta + y\sin\theta$

$y_1 = y\cos\theta - x\sin\theta$

$y_1 = -x\sin\theta + y\cos\theta$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

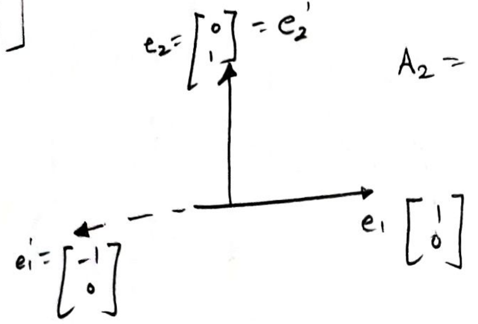


$A_1 \quad (e_1, e_2) \rightarrow 90^\circ \text{ clockwise} \rightarrow (e_1', e_2')$ $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$A_2 \quad (e_1', e_2') \rightarrow \text{mirror} \rightarrow (e_1'', e_2'')$

$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$$A_2 \cdot A_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$



$A_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

what we did was:-

- independently found out for e_1 and e_2 what e_1' and e_2' would be like when 90° clockwise rotation and mirror image is taken.

then we mulipied the matrices to get final result.

Matrix multiplication is not commutative

$\hookrightarrow A_1 \cdot A_2 \neq A_2 \cdot A_1$

however it is associative

$\hookrightarrow A_1 \cdot (A_2 \cdot A_3) = (A_1 \cdot A_2) \cdot A_3$

Gaussian elimination

$$\begin{array}{l} 1. \\ 2. \\ 3. \end{array} \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 15 \\ 21 \\ 13 \end{pmatrix}$$

elimination

3. - 1.

2. - 1.

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 15 \\ 6 \\ -2 \end{pmatrix}$$

back substitution

This is echelon form

A Δ sort of matrix is formed

$c = 2$

$b + c = 6$

$b = 4$

$a + b + 3c = 15$

$a + 4 + 6 = 15$

$a = 5$

another method from previous example:-

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & +1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 15 \\ 6 \\ +2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \\ 2 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix}$$

in this manner,

we get an identity matrix here!

$$A \cdot A^{-1} = A^{-1} \cdot A = I$$

interesting!

Going from gaussian elimination to finding inverse of a matrix

$A^{-1}A = I$

$$\begin{array}{c} 1. \\ 2. \\ 3. \end{array} \quad \overset{A}{\begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{pmatrix}} \overset{B}{\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}} = \overset{I}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

row column

2 = 1. − 2.

3. = 1. − 3.

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & +1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ +1 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} -2 & 0 & 3 \\ -2 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{A} \cdot B = \begin{pmatrix} 0 & -1 & 2 \\ -2 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \longrightarrow A^{-1} \text{ or } B$$

$AB = I$

$B = A^{-1}$

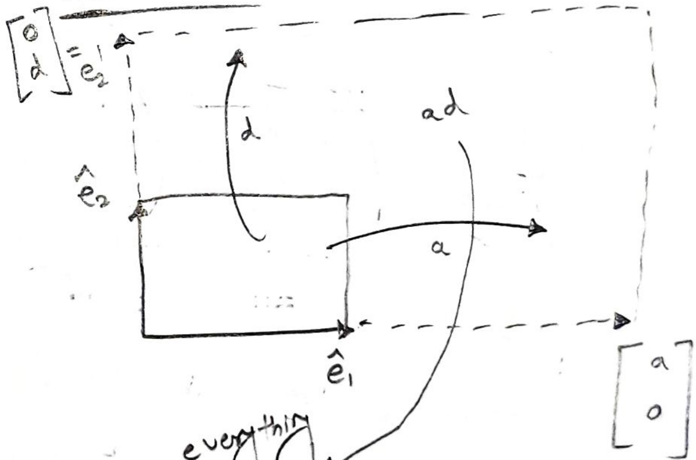
this is called a decomposition process

(we can use this to find the inverse of a matrix for any dimension)

General case

determinant

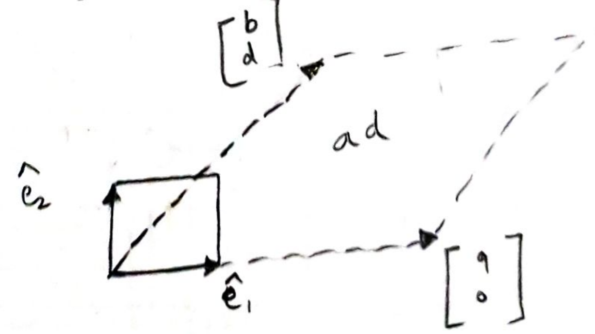
$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$



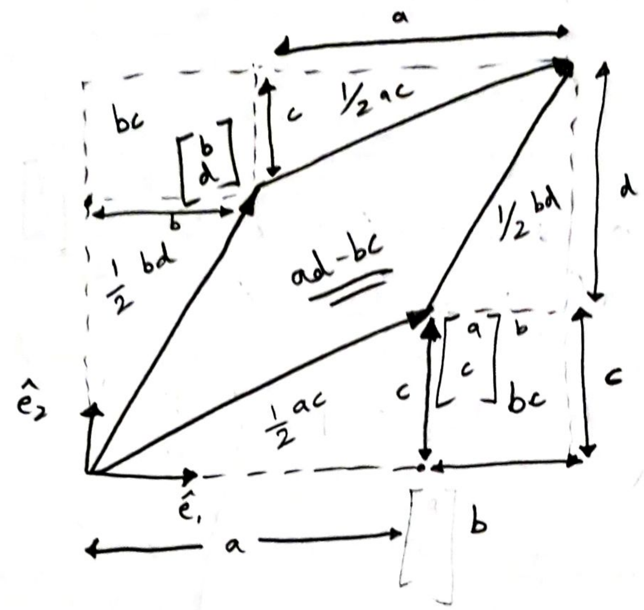
everything is bigger by a factor of **ad**

we call this the determinant of matrix $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$



$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$



area of parallelogram (determinant) $= (a+b)\cdot(c+d)$

$- \frac{1}{2}ac - \frac{1}{2}ac - \frac{1}{2}bd - \frac{1}{2}bd$

$- 2bc$

$= ac + ad + bc + bd$

$- ac - bd - 2bc$

$|A| = ad - bc$

$\frac{1}{ad-bc} \cdot \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}_{A^{-1}} = \underbrace{\begin{pmatrix} \cancel{ad}-bc & 0 \\ 0 & \cancel{ad}-bc \end{pmatrix}}_{I}$

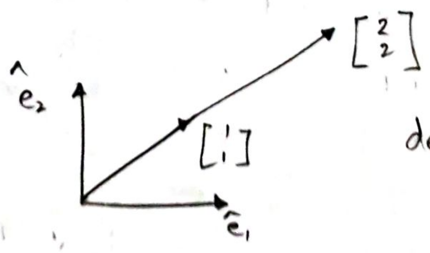
we use the value of det.(A) to get I on the other side

general case → QR decomposition

~~inverting~~

$A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

$|A| = 0$



determinant in this case is 0, area is 0

notice how there is linear dependance here!

∴ only 1 linearly independant basis vector is there.

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 2 & 3 & 7 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ 17 \\ 29 \end{pmatrix}$$

row 3 = row 1 + row 2

col 3 = 2x col 1 + col 2

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ 17 \\ 0 \end{pmatrix}$$

$0c = 0$ (which is sort of true, but now we have infinite solutions for 'c')

(The problem: it is that no new information is gained in the third equation as it is not linearly independant of others (two!))

interesting

"Where the basis vectors describing the matrix aren't linearly independant, then the determinant is 0, we can solve system of simultaneous equations any more"

ALSO

"We can't invert the matrix, as we can't take one over the determinant either"

"We can use a transformation that collapses the number of dimensions in the space but that will come at a cost."

interesting

cost is we won't be able to undo the transformation!

(as information is lost)

Module 4

$$AB = C$$

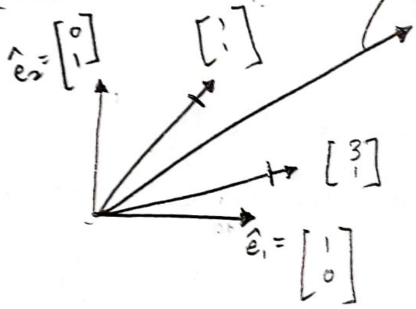
$$C_{ik} = a_{ij} b_{jk}$$

$$\begin{array}{cccc} & 3 & & 4 \\ 2 & \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} & 3 & \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \end{array} = \begin{array}{cc} & 4 \\ 2 & \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \end{array}$$

$$\overset{U}{\begin{pmatrix} U_i \end{pmatrix}} \cdot \overset{V}{\begin{pmatrix} V_i \end{pmatrix}} \xrightarrow[\text{to}]{\text{equivalent}} \begin{bmatrix} U_1 & U_2 & \cdots & U_n \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}$$

$U_i \cdot V_i \longrightarrow$ einstein convention

Matrices changing basis



$\frac{1}{2}\begin{bmatrix} 3 \\ 1 \end{bmatrix} \rightarrow$ wrt basis 2

$\begin{bmatrix} 5 \\ 2 \end{bmatrix} \rightarrow$ wrt basis 1

for (basis 2)

transformation matrix is

$$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$$

wrt basis 1

$$\underset{\substack{\text{trans. matrix} \\ \text{of basis 2} \\ \text{wrt basis 1}}}{\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}} \underset{\substack{\text{vector} \\ \text{wrt} \\ \text{basis 2}}}{\begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}} = \underset{\substack{\text{the same} \\ \text{vector} \\ \text{wrt} \\ \text{basis 1}}}{\begin{bmatrix} 5 \\ 2 \end{bmatrix}}$$

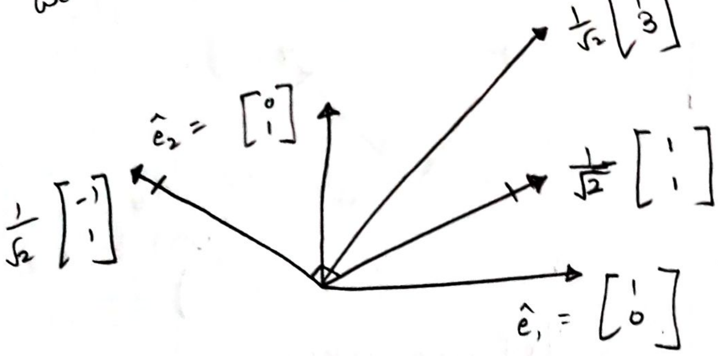
inverse

$\longrightarrow$ to find the transformation matrix of basis 1 wrt basis 2, we take inverse of the matrix (2 wrt 1)

$$\frac{1}{3-1}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$$

$$\underset{\substack{\text{trans. matrix of basis 1} \\ \text{wrt basis 2}}}{\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}} \underset{\substack{\text{the same} \\ \text{vector} \\ \text{wrt} \\ \text{basis 1}}}{\begin{bmatrix} 5 \\ 2 \end{bmatrix}} = \underset{\substack{\text{vector} \\ \text{wrt basis 2}}}{\frac{1}{2}\begin{bmatrix} 3 \\ 1 \end{bmatrix}}$$

"we can use projections if the basis 2 is orthonormal",



transformation matrix of

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix} \xrightarrow{\text{inverse}} (1)\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}$$

(1) → determinant is (1)

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}1\\3\end{bmatrix} = \frac{1}{2}\begin{bmatrix}4\\2\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

trans. matrix of basis 2 wrt 1; vector wrt basis 2; same vector wrt basis 1

alternate :- (using projections)

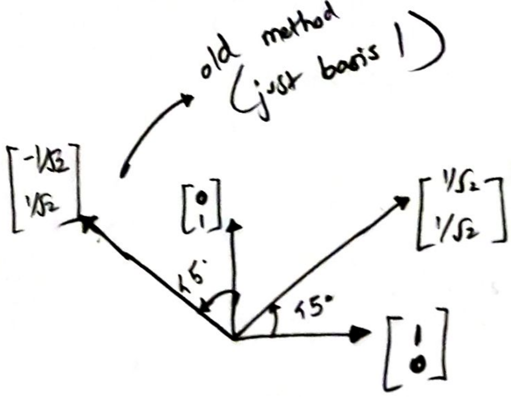
(this happens as they are orthogonal to each other)

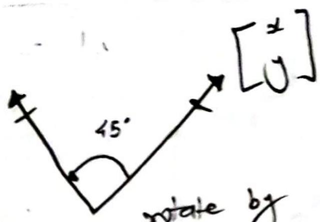
$$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}1\\3\end{bmatrix} = \frac{1}{2}(4) = 2$$

dot product

$$\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}1\\3\end{bmatrix} = \frac{1}{2}(2) = 1$$

$$= \begin{bmatrix}2\\1\end{bmatrix}$$





rotate by 45° wrt basis 2

from last page case:—

basis 2 transformation matrix = $\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$

vector wrt basis 2 = $\begin{bmatrix} x \\ y \end{bmatrix}$

interesting

PROBLEM :— rotate vector wrt basis 2 by 45°

Steps:— (1.) find the vector wrt basis 1 by:-

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(2) rotate it using the old method by 45°

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -1/\sqrt{2} \\ \frac{1}{\sqrt{2}} & 1/\sqrt{2} \end{pmatrix}$$

(3.) Convert it back into a vector wrt basis 2

$$\frac{1}{3-1} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \text{modified } x \\ \text{modified } y \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \text{modified } x \\ \text{modified } y \end{bmatrix}$$

$$\underbrace{B^{-1}RB}_{\text{rotate}} = R_B \text{ for } \begin{bmatrix} x \\ y \end{bmatrix}$$

Transpose of a matrix

$$A^T_{ij} = A_{ji}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

we define

$A_{n \times n}$

$$\left(\begin{pmatrix} \\ a_1 \\ \\ \end{pmatrix} \begin{pmatrix} \\ a_2 \\ \\ \end{pmatrix} \cdots \cdots \begin{pmatrix} \\ a_n \\ \\ \end{pmatrix} \right)$$

interesting

orthonormal

$$\boxed{\begin{array}{lll} a_i \cdot a_j = 0 & i \neq j & \text{(i.e. they are orthogonal)} \\ \quad\;\; = 1 & i = j & \text{(i.e they are of unit length)} \end{array}}$$

dot product

$$A^T = \begin{pmatrix} (& a_1 &) \\ (& a_2 &) \\ & \vdots & \\ (& a_n &) \end{pmatrix}$$

We are assuming $a_i \cdot a_i = 1$ (of unit length)

as $a_1 \cdot a_1 = 1$

$$A^T A = \begin{pmatrix} (& a_1 &) \\ (& a_2 &) \\ & \vdots & \\ (& a_n &) \end{pmatrix} \left(\begin{pmatrix} \\ a_1 \\ \\ \end{pmatrix} \begin{pmatrix} \\ a_2 \\ \\ \end{pmatrix} \cdots \begin{pmatrix} \\ a_n \\ \\ \end{pmatrix} \right) = \underbrace{\begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ 0 & 0 & \cdots & 0 \; 0 \; 1 \end{pmatrix}}_{I}$$

from above

for orthonormal matrix

$$A^T = A^{-1} \quad (\text{i.e. } A^T A = I)$$

proof for orthonormal set to be linearly independent
(by contradiction)

$B = \{ \vec{V_1}, \vec{V_2}, \dots \vec{V_k} \}$

assume:- All the vectors in B have length 1

i.e. $|V_i| = 1$ for $i = 1, 2, \dots k$

i.e $V_i \cdot V_i = 1$

also: all the vectors are orthogonal to each other

$$V_i \cdot V_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

B is an orthonormal set

$\hookrightarrow$ B is linearly independant

$\downarrow$ PROOF

assume B is linearly dependant

we know for i, j where $i \neq j$

$\vec{V_i}, \vec{V_j} \in B$

and $\vec{V_i} \cdot \vec{V_j} = 0$

assuming linear dependance:-

$\vec{V_i} = c\vec{V_j}$

$c\vec{V_j} \cdot \vec{V_j} = c(\vec{V_j} \cdot \vec{V_j}) = c(1)$

$= \underline{\underline{c}}$

$\therefore$ we reach a contradiction!

eg.- $\vec{V_1} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$ $\vec{V_2} = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

$$B = \{\vec{V_1}, \vec{V_2}\}$$

$$|\vec{V_1}| = \vec{V_1} \cdot \vec{V_1} = \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = \frac{9}{9} = 1$$

$$|\vec{V_2}| = \vec{V_2} \cdot \vec{V_2} = \frac{4}{9} + \frac{1}{9} + \frac{4}{9} = 1$$

$$V_1 \cdot V_2 = \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0$$

$\therefore$ they are a part of orthonormal basis vector set V

$$V = \text{span}(\vec{V_1}, \vec{V_2})$$

interesting

Gram–Schmidt process

$$V = \{v_1, v_2, v_3 \, \ldots \ldots V_n\}$$

↳ To construct an orthonormal basis vectors for V

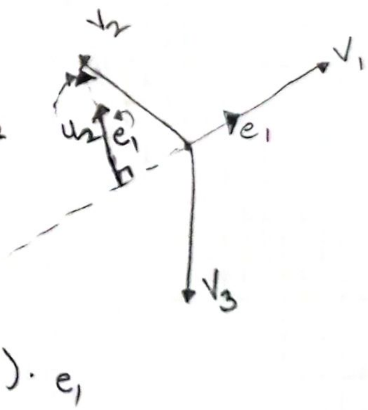
for first basis vector $e_1 = \dfrac{V_1}{|V_1|}$ (we have normalized V_1 to get a unit length)

this term is projection of V_2 on e_1

for v_2 :-

$$V_2 = \overbrace{\frac{V_2 \cdot e_1}{|e_1|} \cdot e_1}^{} + u_2$$

(but $|e_1| = 1$)



$$u_2 = V_2 - \frac{(V_2 \cdot e_1) \cdot e_1}{|e_1|}$$

↓

we can then take normalized version of U_2 to get:- $\dfrac{u_2}{|u_2|} = e_2$ (which will be orthogonal to e_1 and both of unit length) for V_3:-

(by projection on e_1 and e_2)

$$u_3 = V_3 - \frac{(V_3 \cdot e_1)}{|e_1|} \cdot e_1 - \frac{(V_3 \cdot e_2)}{|e_2|} \cdot e_2 \ldots$$

(where $|e_1| = |e_2| = |e_3| = \ldots = 1$)

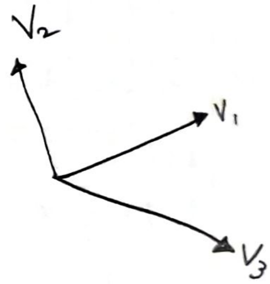
and $e_3 = \dfrac{u_3}{|u_3|}$

Reflecting in a plane

Applications:- "this can be useful to ~~transform~~ images and faces for the purpose of facial recognition."

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \quad \begin{pmatrix}2\\0\\1\end{pmatrix} \quad \begin{pmatrix}3\\1\\-1\end{pmatrix}$$

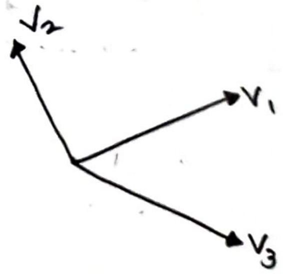
$V_1 \qquad V_2 \qquad V_3$



$$e_1 = \frac{V_1}{|V_1|} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \quad \begin{pmatrix}2\\0\\1\end{pmatrix} \quad \begin{pmatrix}3\\1\\-1\end{pmatrix}$$

$V_1 \qquad V_2 \qquad V_3$



→ all 3 are in different planes

$$e_1 = \frac{V_1}{|V_1|} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$u_2 = V_2 - (V_2 \cdot e_1) \cdot e_1 = \begin{pmatrix}2\\0\\1\end{pmatrix} - \left[\begin{pmatrix}2\\0\\1\end{pmatrix} \cdot \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}\right] \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$e_2 = \frac{u_2}{|u_2|} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

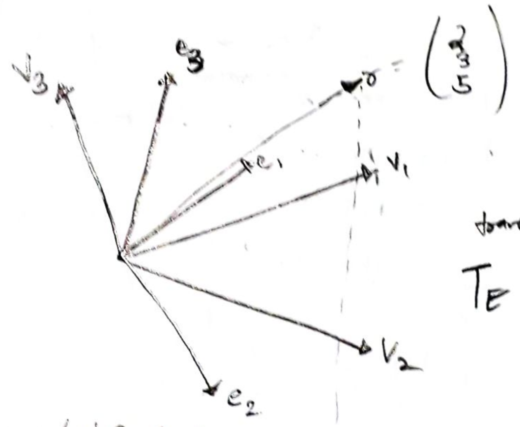
$u_3 = v_3 - (v_3 \cdot e_1)\cdot e_1 - (v_3 \cdot e_2)\cdot e_2$

$$= \begin{pmatrix}3\\1\\-1\end{pmatrix} - \left[\begin{pmatrix}3\\1\\-1\end{pmatrix}\frac{1}{\sqrt3}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\right]\cdot\frac{1}{\sqrt3}\begin{pmatrix}1\\1\\1\end{pmatrix} - \left[\begin{pmatrix}3\\1\\-1\end{pmatrix}\cdot\frac{1}{\sqrt2}\begin{pmatrix}1\\-1\\0\end{pmatrix}\right]\frac{1}{\sqrt2}\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$= \begin{bmatrix}3\\1\\-1\end{bmatrix} - \frac{1}{\cancel{3}}\cdot\cancel{3}\begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{1}{\cancel{2}}\cdot\cancel{2}\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{bmatrix}1\\1\\-2\end{bmatrix}$$

$$e_3 = \frac{1}{\sqrt6}\begin{pmatrix}1\\1\\-2\end{pmatrix}$$

$$E = \begin{bmatrix}\begin{pmatrix}e_1\end{pmatrix} & \begin{pmatrix}e_2\end{pmatrix} & \begin{pmatrix}e_3\end{pmatrix}\end{bmatrix}$$



r' (reflection)

transformation matrix

$$T_E = \begin{array}{ccc} \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & -1\end{pmatrix} \\ e_1 \quad e_2 \quad e_3\end{array}$$

in wrt original basis (not e_1, e_2, e_3)

$$\begin{tikzcd} \boxed{r} \arrow[r, "\text{hard}"] \arrow[d, "E^{-1}"'] & \boxed{r'} \\ \boxed{r_E} \arrow[r, "T_E"'] & \boxed{r'_E} \arrow[u, "E"'] \end{tikzcd}$$

Also as E is orthonormal

$$E^T = E^{-1}$$

$E T_E E^{-1} r = r'$ converted to $\therefore E T_E E^T r = r'$

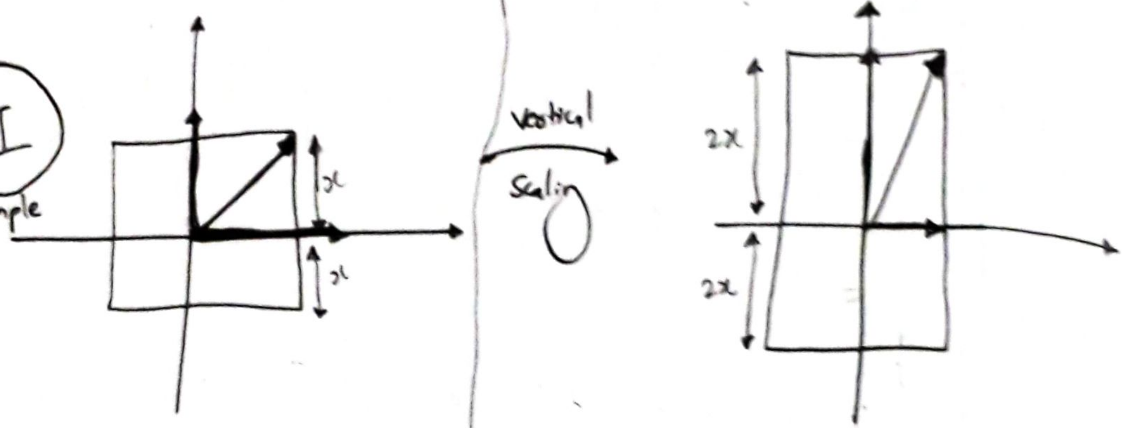
solving gives $r' = \frac{1}{3}\begin{pmatrix}11\\14\\5\end{pmatrix}$

Module 5

eigen values and eigenvectors

"eigen ≈ characteristic"

(I) example



diagonal vector changes its direction

as these values are characteristic of this particular transform, we refer to them as **eigen vectors**

horizontal and vertical vectors do not change their direction. (SPAN) (although vertical vector gets scaled)

also as vertical vector doubles its value it has eigen value of 2

horizontal vector remains same

it has eigen value of 1

We take a transformation

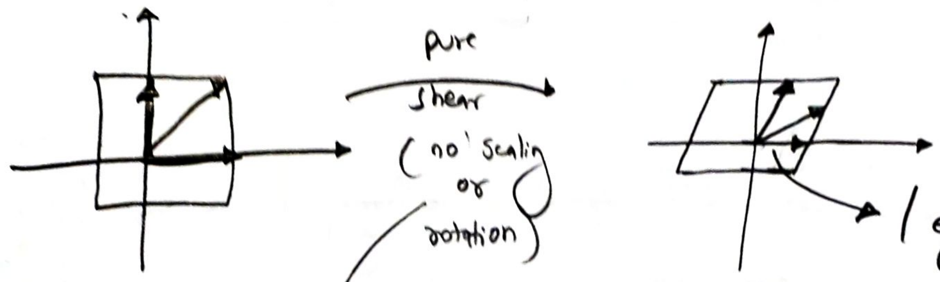
↓

we look for vectors who are still laying on the same span as before (eigenvectors)

↓

we measure how much their length has changed (eigenvalues)

example



1 eigenvector ($\hat{x}$ axis one)

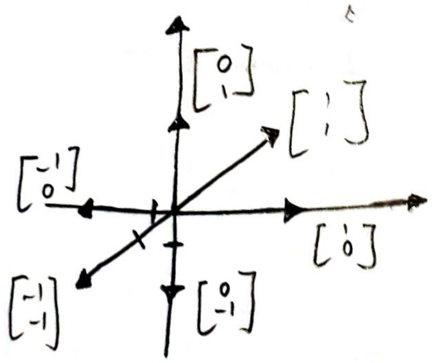
area remains same

interesting

vectors $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

mirrored using $T = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

new is represented by $\longrightarrow$

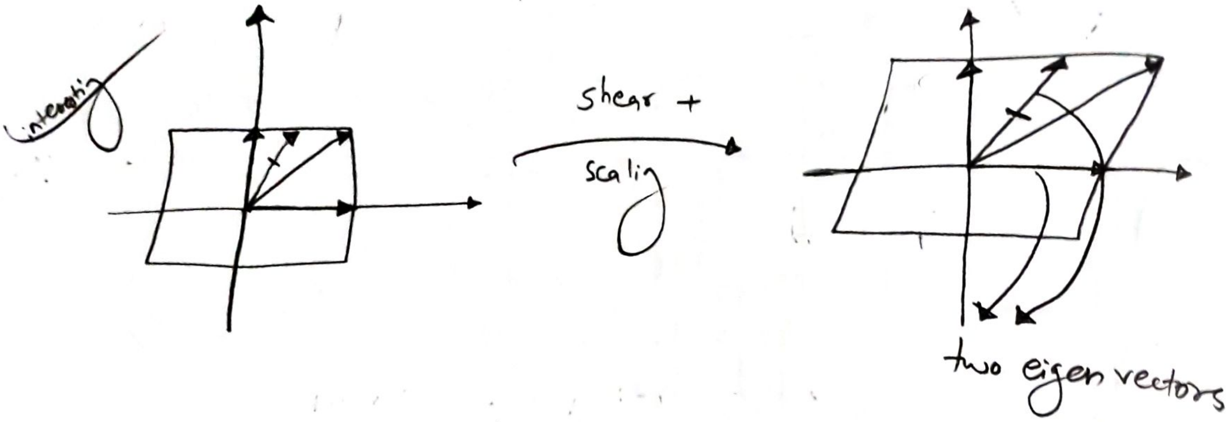


(as span remains the same, all 3 are eigenvectors with eigenvalues of -1)

↓

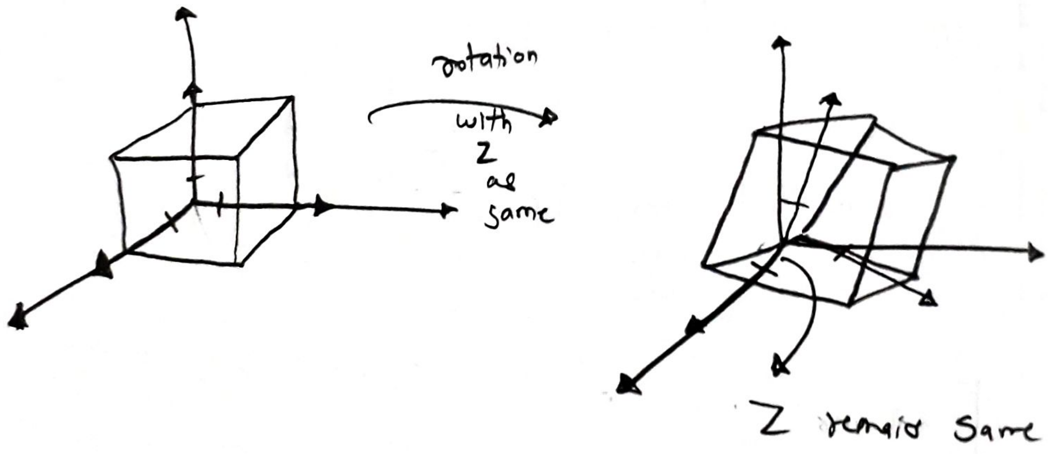
interesting

The above situation can also be arrived by performing $180°$ rotation.



∴ they are not easy to spot!

" In 3-D space, performing rotation and find eigenvectors also tells us as a physical interpretation, that if we find the eigenvector of 3D rotation, it also means we've found the axis of rotation"



Calculating eigenvectors

$x \rightarrow$ vector of length 'n'

$A \rightarrow$ matrix of size '$n \times n$'

$$Ax = \lambda x$$

$\lambda \rightarrow$ some scalar value like $(3, 2, -1/3 \ldots)$

$$(A - \lambda I)x = 0$$

$\downarrow$

as we cannot subtract matrix with scalar,
we multiply λ with Identity matrix
and subtract.

if $x = 0$, trivial solution
(it indicates, it is dimentionless)

if $x \neq 0$,
we have to make $(A - \lambda I) \rightarrow 0$

if we get
$$\det(A - \lambda I) = 0$$
we can get $A - \lambda I = 0$

$\therefore$ the problem reduces to :-
$$\det(A - \lambda I) = 0$$

eg:- for $2 \times 2 \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = \lambda^2 - (a+d)\lambda + ad - bc = 0$$

eg. $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ $\det \begin{pmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix}$

I example

$(A - \lambda I)x = 0$

$(1-\lambda)(2-\lambda) = 0$

$\lambda = 1, \lambda = 2$

@ $\lambda = 1$: $\begin{pmatrix} 1-1 & 0 \\ 0 & 2-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = 0$

$\boxed{x_2 = 0}$

@ $\lambda = 2$: $\begin{pmatrix} 1-2 & 0 \\ 0 & 2-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$= \begin{pmatrix} -x_1 \\ 0 \end{pmatrix} = 0$

$\boxed{x_1 = 0}$

$\therefore$ @ $\lambda = 1$: $x = \begin{pmatrix} t \\ 0 \end{pmatrix}$ as $x_2 = 0$

where t is any value (real)

$\therefore$ @ $\lambda = 2$: $x = \begin{pmatrix} 0 \\ t \end{pmatrix}$ as $x_1 = 0$

{ If we get imaginary roots :- no eigenvectors are found in real space interesting We do not consider a zero vector $\begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix}$ as an eigenvector of a matrix.

Changing to the eigenbasis

$$T^n = \begin{pmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{pmatrix}$$

if $V_0 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$ $\quad T = \begin{bmatrix} 0.9 & 0.8 \\ -1 & 0.35 \end{bmatrix}$ → transformation matrix

$$V_n = T^n V_0$$

→ if n is very large, the process can be tedious

however for a diagonal matrix like this process becomes easier!

" so if we get T, which is not a diagonal matrix, we change to a basis where T becomes diagonal (called eigen-basis). "

if T is the transformation matrix:— (eg:- $T = \begin{bmatrix} 0.9 & 0.8 \\ -1 & 0.35 \end{bmatrix}$)

$$C = \begin{pmatrix} V_1 & V_2 & V_3 \\ \vdots & \vdots & \vdots \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

→ eigen vectors

($\lambda_1, \lambda_2, \lambda_3 \in$ eigen values)

$$T = CDC^{-1}$$

$$T^2 = CDC^{-1}CDC^{-1} = CD^2C^{-1}$$

$$\boxed{T^n = CD^nC^{-1}}$$